

Spacecraft Position and Attitude Formation Control using Line-of-Sight Observations

Tse-Huai Wu and Taeyoung Lee*

Abstract—This paper studies formation control of an arbitrary number of spacecraft based on a serial network structure. The leader controls its absolute position and absolute attitude with respect to an inertial frame, and the followers control its relative position and attitude with respect to another spacecraft assigned by the serial network. The unique feature is that both the absolute attitude and the relative attitude control systems are developed directly in terms of the line-of-sight observations between spacecraft, without need for estimating the full absolute and relative attitudes, to improve accuracy and efficiency. Control systems are developed on the nonlinear configuration manifold, guaranteeing exponential stability. Numerical examples are presented to illustrate the desirable properties of the proposed control system.

I. INTRODUCTION

Spacecraft formation flight has been intensively studied as distributing tasks over a group of low-cost spacecraft is more efficient and robust than operating a single large and powerful spacecraft [1]. For cooperative spacecraft missions, precise control of relative configurations among spacecraft is critical for success. For interferometer missions like Darwin, spacecraft in formation should maintain specific relative position and relative attitude configurations precisely. Precise relative position control and estimation have been addressed successfully, for example, by utilizing carrier-phase differential GPS [2], [3].

For relative attitude control, there have been various approaches, including leader-follower strategy [4], [5], behavior-based controls [6], [7] and virtual structures [8], [9]. These approaches have distinct features, but there is a common framework: the absolute attitude of each spacecraft in formation is determined independently and individually by using an on-board sensor, such as inertial measurement units and star trackers, and they are transmitted to other spacecraft to determine relative attitude between them. As the relative attitudes are determined *indirectly* by comparing absolute attitudes, there is a fundamental limitation in accuracies. More explicitly, measurement and estimation errors of multiple sensors are accumulated in determination of the relative attitudes.

Vision-based sensors have been widely applied for navigation of autonomous vehicles, and recently, they are proposed for determination of relative attitudes. It is shown that the line-of-sight (LOS) measurements between two spacecraft

determine the relative attitude between them completely, and based on it, an extended Kalman filter is developed [10], [11]. Recently, these are also utilized in stabilization of relative attitude between two spacecraft [12], and tracking control of relative attitude formation between multiple spacecraft [13], [14], where control inputs are directly expressed in terms of line-of-sight measurements, without need for constructing the full, absolute attitude or the relative attitudes.

However, these prior results are restrictive in the sense that the relative positions among spacecraft, and therefore the lines-of-sight with respect to the inertial frame, are assumed to be fixed during the whole attitude maneuvers. Therefore, they cannot be applied to the cases where both the relative positions and the relative attitudes should be controlled concurrently at the similar time scale.

The objective of this paper is to eliminate such restrictions. In this paper, the translational dynamics and the rotational dynamics of each spacecraft are considered, and a serial network structure is defined. The first spacecraft at the network, namely the leader controls its absolute position and absolute attitude with respect to an inertial frame, and the remaining spacecraft, namely followers control its relative position and relative attitude with respect to another spacecraft ahead in the serial chain of network. The main contribution is that both the absolute attitude controller of the leader, and the relative attitude controller of the followers are defined directly in terms of the line-of-sight measurements, and exponential stability is guaranteed without the restrictive assumption that the relative positions are fixed.

Therefore, the control system proposed in this paper inherits the desirable features of the relative attitude controls based on the lines-of-sight [12], [13], [14], namely low cost and long-term stability, while requiring no corrections in measurements as opposed to gyros, or eliminating needs for computationally expensive star tracking algorithms. But, it can be applied to more realistic cases where the relative positions are controlled simultaneously. Another distinct feature of this paper is that the absolute attitude of the leader is also controlled, whereas the preliminary works are only focused on relative attitude formation control [12]. All of these are constructed on the special orthogonal group to avoid singularities, complexities associated with local parameterizations or ambiguities of quaternions in representing attitudes.

II. PROBLEM FORMULATION

Consider n spacecraft, where each spacecraft is modeled as a rigid body. Define an inertial frame, and the body-fixed frame for each spacecraft. The configuration of the

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i -th spacecraft is defined by $(R_i, x_i) \in \text{SE}(3)$, where the special Euclidean group $\text{SE}(3)$ is the semi-direct product of the special orthogonal group $\text{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \det[R] = 1\}$, and \mathbb{R}^3 . The rotation matrix $R_i \in \text{SO}(3)$ represents the linear transformation of the representation of a vector from the i -th body fixed frame to the inertial frame, and the vector $x_i \in \mathbb{R}^3$ denote the location of the mass center of the i -th spacecraft with respect to the inertial frame.

A. Dynamic Model

Let $m_i \in \mathbb{R}$ and $J_i \in \mathbb{R}^{3 \times 3}$ be the mass and the inertia matrix of the i -th spacecraft. The equations of motion are given by

$$m_i \ddot{x}_i = f_i, \quad (1)$$

$$\dot{x}_i = v_i, \quad (2)$$

$$J_i \dot{\Omega}_i + \Omega_i \times J_i \Omega_i = u_i, \quad (3)$$

$$\dot{R}_i = R_i \hat{\Omega}_i, \quad (4)$$

where $v_i, \Omega_i \in \mathbb{R}^3$ are the translational velocity and the rotational angular velocity of the i -th spacecraft, respectively. The control force acting on the i -th spacecraft is denoted by $f_i \in \mathbb{R}^3$, and the i -th control moment is denoted by $M_i \in \mathbb{R}^3$. The vectors v_i, f_i are represented with respect to the inertial frame, and Ω_i, M_i are represented with respect to the i -th body-fixed frame.

The *hat* map $\wedge : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ transforms a vector in \mathbb{R}^3 to a 3×3 skew-symmetric matrix such that $\hat{x}y = (x)^\wedge y = x \times y$ for any $x, y \in \mathbb{R}^3$. The inverse of the hat map is denoted by the *vee* map $\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$. A few properties of the hat map are summarized as follows:

$$\widehat{x \times y} = \hat{x}\hat{y} - \hat{y}\hat{x} = yx^T - xy^T, \quad (5)$$

$$\text{tr}[\hat{x}A] = \frac{1}{2}\text{tr}[\hat{x}(A - A^T)] = -x^T(A - A^T)^\vee, \quad (6)$$

$$\hat{x}A + A^T\hat{x} = (\{\text{tr}[A]I_{3 \times 3} - A\}x)^\wedge, \quad (7)$$

$$R\hat{x}R^T = (Rx)^\wedge, \quad (8)$$

$$\exp(\widehat{Ry}) = R \exp(\hat{y}) R^T, \quad (9)$$

for any $x, y \in \mathbb{R}^3$, $A \in \mathbb{R}^{3 \times 3}$, and $R \in \text{SO}(3)$.

Suppose that n spacecraft are serially connected by daisy-chaining. For notational convenience, it is assumed that spacecraft indices are ordered along the serial network. The first spacecraft of the serial chain, namely Spacecraft 1 is selected to be the *leader*, and its absolute position and its absolute attitude are controlled with respect to the inertial frame. The remaining spacecraft are considered as *followers*, where each follower controls its relative position and its relative attitude with respect to the spacecraft that is one step ahead in the serial chain. For example, Spacecraft 2 controls its relative configuration with respect to Spacecraft 1.

It can be shown that the controller structure of each of followers are identical. To make the subsequent derivations more concrete and concise, the control systems are defined for the leader, Spacecraft 1, and the first follower, Spacecraft 2, only. Later, it is generalized for control systems of the remaining spacecraft.

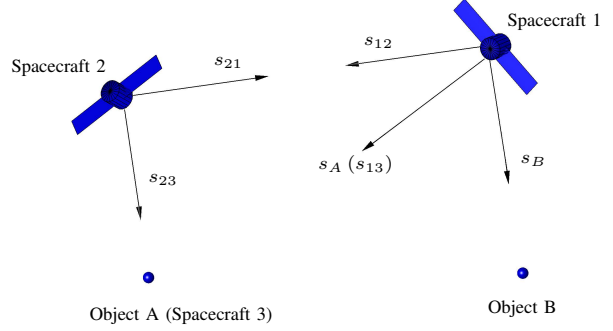


Fig. 1. Line-of-sight (LOS) Measurements: The leader, Spacecraft 1, measures the LOS toward to distinct objects A, B to control its absolute attitude (the object A is selected as Spacecraft 3). To control the relative attitude between Spacecraft 1 and Spacecraft 2, they measure the LOS toward each other, and also toward the common object selected as Spacecraft 3.

Each spacecraft is assumed to measure the line-of-sight toward two assigned objects via vision-based sensors to control its attitude. Line-of-sight measurements for the leader and the followers are described as follows.

B. Line-of-Sight Measurements of Leader

Suppose that there are two distinct objects, namely A, B , such as distant stars, whose locations in the inertial reference frame are available. Let $s_A, s_B \in S^2 = \{q \in \mathbb{R}^3 \mid \|q\| = 1\}$ be the unit-vectors representing the directions from Spacecraft 1 to the object A and to the object B , respectively. The vectors s_A and s_B are expressed with respect to the inertial frame, and we have $s_A \times s_B \neq 0$, as they are distinct objects. The time-derivatives of these unit-vectors are given by

$$\dot{s}_A = \mu_A \times s_A, \quad \dot{s}_B = \mu_B \times s_B, \quad (10)$$

where $\mu_A, \mu_B \in \mathbb{R}^3$ are angular velocities of s_A, s_B , respectively, that can be determined explicitly by the relative translational motion of the leader with respect to the objects A, B .

Assume the leader is equipped with a vision-based sensor that can measure the directions toward the objects A and B . These two line-of-sight measurements are expressed with respect to the first body-fixed frame, and they are defined as $b_A, b_B \in S^2$. Note that s_A, s_B and b_A, b_B are related by the rotation matrix, i.e.,

$$s_j = R_1 b_j \text{ or equivalently, } b_j = R_1^T s_j, \quad (11)$$

for $j \in \{A, B\}$. From (10) and (11), we can obtain the kinematic equations for b_A, b_B as

$$\begin{aligned} \dot{b}_j &= \dot{R}_1^T s_j + R_1^T \dot{s}_j = -\hat{\Omega}_1 R_1^T s_j + R_1^T \hat{\mu}_j s_j \\ &= -\hat{\Omega}_1 b_j + R_1^T \hat{\mu}_j R_1 R_1^T s_j = b_j \times (\Omega_1 - R_1^T \mu_j), \end{aligned} \quad (12)$$

for $j \in \{A, B\}$.

C. Line-of-Sight Measurements of Follower

Spacecraft 2 controls its relative attitude and its relative position with respect to Spacecraft 1. The relative configurations are defined as follows. The relative attitude of Spacecraft 2 with respect to Spacecraft 1 is represented by a rotation matrix $Q_{21} \in \text{SO}(3)$, given by

$$Q_{21} = R_1^T R_2. \quad (13)$$

From (4), the time-derivative of Q_{21} is given by

$$\begin{aligned} \dot{Q}_{21} &= -\hat{\Omega}_1 R_1^T R_2 + R_1^T R_2 \hat{\Omega}_2 = Q_{21} \hat{\Omega}_2 - \hat{\Omega}_2 Q_{21} \\ &= Q_{21} (\Omega_2 - Q_{21}^T \Omega_1)^\wedge \triangleq Q_{21} \hat{\Omega}_{21}, \end{aligned} \quad (14)$$

where the relative angular velocity vector of Spacecraft 2 with respect to Spacecraft 1 is defined as $\Omega_{21} = \Omega_2 - Q_{21}^T \Omega_1 \in \mathbb{R}^3$. The relative position of Spacecraft 2 with respect to Spacecraft 1 is given by

$$x_{21} = x_2 - x_1. \quad (15)$$

To control the relative attitude between the pair of Spacecraft 1 and Spacecraft 2, a third common object, namely Spacecraft 3 is assigned to form a triangular structure. It is assumed that Spacecraft 1 and Spacecraft 2 measure the line-of-sight toward each other, and also toward Spacecraft 3. Furthermore, Spacecraft 3 does not lie on the line joining Spacecraft 1 and 2.

Let $s_{ij} \in \mathbb{S}^2$ be the unit-vector from the i -th spacecraft toward the j -th spacecraft, i.e., $s_{ij} = \frac{x_j - x_i}{\|x_j - x_i\|}$, and let $b_{ij} \in \mathbb{S}^2$ be the line-of-sight measurement from the i -th spacecraft toward the j -th spacecraft, represented with respect to the i -th body-fixed frame, i.e., $s_{ij} = R_i b_{ij}$. According to the above assumptions, we have $s_{13} \times s_{23} \neq 0$, and there are four measurements $\{b_{12}, b_{13}, b_{21}, b_{23}\}$ for the pair of Spacecraft 1 and 2.

It has been shown that the relative attitude Q_{21} can be completely determined from the following constraints [12]:

$$b_{12} = -Q_{21} b_{21}, \quad (16)$$

$$b_{123} = -Q_{21} b_{213}, \quad (17)$$

where $b_{123} = \frac{b_{12} \times b_{13}}{\|b_{12} \times b_{13}\|} \in \mathbb{S}^2$ and $b_{213} = \frac{b_{21} \times b_{23}}{\|b_{21} \times b_{23}\|} \in \mathbb{S}^2$. The first constraint (16) states that the unit vector from Spacecraft 1 to Spacecraft 2 is opposite to the unit vector from Spacecraft 2 to Spacecraft 1, i.e., $s_{12} = -s_{21}$. The second constraint (17) implies that the plane spanned by s_{12} and s_{13} should be co-planar with the plane spanned by s_{21} and s_{23} . For given LOS measurements $\{b_{12}, b_{13}, b_{21}, b_{23}\}$, the relative attitude Q_{21} is uniquely determined by solving (16) and (17) for Q_{21} .

Similar to the definitions in (11)-(12), we have

$$s_{ij}(t) = R_i(t) b_{ij}(t), \quad b_{ij}(t) = R_i(t)^T s_{ij}(t), \quad (18)$$

$$\dot{s}_{ij} = \mu_{ij} \times s_{ij}, \quad (19)$$

$$\dot{b}_{ij} = b_{ij} \times (\Omega_i - R_i^T \mu_{ij}), \quad (20)$$

for $(i, j) \in \{(1, 2), (1, 3), (2, 1), (2, 3)\}$. Further, for $(i, j, k) \in \{(1, 2, 3), (2, 1, 3)\}$, we have

$$\dot{s}_{ijk} \triangleq \mu_{ijk} \times s_{ijk}, \quad (21)$$

where $s_{ijk} = \frac{s_{ij} \times s_{ik}}{\|s_{ij} \times s_{ik}\|} = R_i b_{ijk} \in \mathbb{S}^2$ and $\mu_{ijk} \in \mathbb{R}^3$ is the angular velocity of s_{ijk} . Similar to (20), we can write

$$\dot{b}_{ijk} = \dot{R}_i^T s_{ijk} + R_i^T \dot{s}_{ijk} = b_{ij} \times (\Omega_i - R_i^T \mu_{ijk}). \quad (22)$$

Notice that $\|b_{ij} \times b_{ik}\| = \|R_i^T s_{ij} \times R_i^T s_{ik}\| = \|s_{ij} \times s_{ik}\|$, and it is non-zero.

D. Spacecraft Formation Control Problem

Next, we define formation control problem. For the leader, Spacecraft 1, the desired *absolute* attitude trajectory $R_1^d(t) \in \text{SO}(3)$ is given, and it satisfies the following kinematic equation

$$\dot{R}_1^d = R_1^d \hat{\Omega}_1^d, \quad (23)$$

where $\Omega_1^d \in \mathbb{R}^3$ is the desired angular velocity of Spacecraft 1. We transform this into the desired line-of-sight and its angular velocity as follows. The corresponding desired line-of-sight measurements are given by

$$b_i^d = (R_1^d)^T s_i, \quad i \in \{A, B\}. \quad (24)$$

Similar with (12), the kinematic equations for desired line-of-sight measurements can be obtained as

$$\dot{b}_i^d = b_i^d \times (\Omega_1^d - R_1^d \mu_i), \quad i \in \{A, B\}. \quad (25)$$

The desired position and the desired velocity of Spacecraft 1 are given by $x_1^d(t), v_1^d(t) \in \mathbb{R}^3$.

For the follower, Spacecraft, the desired *relative* attitude trajectory is defined as $Q_{21}^d(t) \in \text{SO}(3)$, and it satisfies the following kinematics equation,

$$\dot{Q}_{21}^d = Q_{21}^d \hat{\Omega}_{21}^d, \quad (26)$$

where $\hat{\Omega}_{21}^d \in \mathbb{R}^3$ is the desired relative angular velocity vector, satisfying

$$\Omega_{21}^d = \Omega_2^d - Q_{21}^{d\top} \Omega_1^d. \quad (27)$$

Let the desired relative position and the relative velocity of Spacecraft 2 be $x_{21}^d(t), v_{21}^d(t) \in \mathbb{R}^3$, and they satisfy

$$\dot{x}_{21}^d = v_{21}^d = v_2^d - v_1^d. \quad (28)$$

Assumption 1: The magnitude of each desired angular velocity is bounded by

$$\|\Omega^d\| \leq B_{\Omega_d}, \quad (29)$$

where $\Omega^d \in \{\Omega_{21}^d, \Omega_2^d, \Omega_1^d\}$.

Assumption 2: The magnitude of angular velocity of each line-of-sight measurement is bounded by

$$\|\mu\| \leq B_\mu, \quad (30)$$

where $B_\mu \in \mathbb{R}^+$ is a positive constant. The μ here includes $\{\mu_A, \mu_B, \mu_{12}, \mu_{21}, \mu_{123}, \mu_{213}\}$.

The first assumption is natural in any tracking problem, and the second assumption is satisfied if there is no collision between spacecraft and the velocities of each spacecraft are bounded.

The goal is to design control inputs (u_1, u_2, f_1, f_2) such that the zero equilibrium of the position and attitude tracking errors becomes asymptotically stable, and in particular, it is required the the control moments u_1, u_2 are directly expressed in terms of the line-of-sight measurements.

III. ATTITUDE AND POSITION TRACKING ERROR VARIABLES

As a preliminary work before designing control systems, in this section, we present error variables for each of absolute attitude tracking error, relative attitude tracking error, and position tracking error.

A. Absolute Attitude Error Variables

The fundamental idea of constructing attitude control system in terms of LOS measurements is utilizing the property that the attitude error becomes zero if the LOS are aligned to their desired values, i.e., we have $R_1 = R_1^d$ if $b_A = b_A^d$ and $b_B = b_B^d$ [14]. This motivates the following definition of the configuration error function for each line-of-sight:

$$\Psi_j = 1 - b_j \cdot b_j^d = 1 - R_1^T s_j \cdot R_1^{dT} s_j, \quad (31)$$

for each object $j \in \{A, B\}$. They are combined into

$$\Psi_1 = k_{b_A} \Psi_A + k_{b_B} \Psi_B, \quad (32)$$

where $k_{b_A} \neq k_{b_B} > 0$ are positive constants. The error function Ψ_i refers to the corresponding error of LOS measurements, and Ψ_1 represents the combined errors for Spacecraft 1.

By finding the derivatives of the configuration error functions, we obtain the configuration error vectors as follows.

$$e_{b_A} = b_A \times b_A^d, \quad e_{b_B} = b_B \times b_B^d, \quad (33)$$

$$e_b = k_{b_A} e_{b_A} + k_{b_B} e_{b_B}. \quad (34)$$

Additionally, the angular velocity error vector is defined as

$$e_{\Omega_1} = \Omega_1 - \Omega_1^d. \quad (35)$$

It can be shown that the above error variables satisfy the following properties.

Proposition 1: (i) The error function Ψ_1 and the error vector e_b can be rewritten as

$$\Psi_1 = \text{tr}[K_1(I_{3 \times 3} - R_1 R_1^{dT})], \quad (36)$$

$$e_b = [R_1^{dT} K_1 R_1 - R_1^T K_1 R_1^d]^V, \quad (37)$$

where $K_1 = k_{b_A} s_A s_A^T + k_{b_B} s_B s_B^T \in \mathbb{R}^{3 \times 3}$ is symmetric and $I_{3 \times 3}$ is the 3 by 3 identity matrix.

(ii) Ψ_1 is locally quadratic, more explicitly,

$$\underline{\psi}_b \|e_b\|^2 \leq \Psi_1 \leq \bar{\psi}_b \|e_b\|^2, \quad (38)$$

where $\underline{\psi}_b = \frac{h_1}{h_2 + h_3}$ and $\bar{\psi}_b = \frac{h_1 h_4}{h_5(h_1 - \psi_1)}$ for

$$\begin{aligned} h_1 &= 2 \min\{k_{b_A}, k_{b_B}\}, \\ h_2 &= 4 \max\{(k_{b_A} - k_{b_B})^2, k_{b_A}^2, k_{b_B}^2\}, \\ h_3 &= 4(k_{b_A} + k_{b_B})^2, \\ h_4 &= 2(k_{b_A} + k_{b_B}), \\ h_5 &= 4 \min\{k_{b_A}^2, k_{b_B}^2\}, \end{aligned}$$

and ψ_1 is a positive constant satisfying $\Psi_1 < \psi_1 < h_1$.

(iii) $\|e_b\| \leq k_{b_A} + k_{b_B}$.

(iv) $\frac{d}{dt} \Psi_1 = e_b \cdot e_{\Omega_1} + \Gamma \|e_b\|$.

(v) $\|\frac{d}{dt} e_b\| \leq \frac{1}{\sqrt{2}}(k_{b_A} + k_{b_B}) \|e_{\Omega_1}\| + (B_{\Omega_d} + B_1) \|e_b\|$.

Proof: See Appendix, Section B. ■

B. Relative Attitude Error Variables

Similarly, we define error variables for the relative attitude. It is also based on the fact that we have $Q_{21} = Q_{21}^d$ provided that $b_{12} = -Q_{21}^d b_{21}$ and $b_{123} = -Q_{21}^d b_{213}$ [13]. Configuration error functions that represent the errors in satisfaction of (16) and (17) are defined as

$$\begin{aligned} \Psi_{21}^\alpha &= \frac{1}{2} \|b_{12} + Q_{21}^d b_{21}\|^2 = 1 + b_{12} \cdot Q_{21}^d b_{21}, \\ \Psi_{21}^\beta &= 1 + b_{123} \cdot Q_{21}^d b_{213}. \end{aligned} \quad (39)$$

For positive constants $k_{21}^\alpha \neq k_{21}^\beta$, these are combined into

$$\Psi_{21} = k_{21}^\alpha \Psi_{21}^\alpha + k_{21}^\beta \Psi_{21}^\beta. \quad (40)$$

These yield the configuration error vectors as

$$\begin{aligned} e_{21}^\alpha &= (Q_{21}^{dT} b_{12}) \times b_{21}, \quad e_{21}^\beta = (Q_{21}^{dT} b_{123}) \times b_{213}, \\ e_{21} &= k_{21}^\alpha e_{21}^\alpha + k_{21}^\beta e_{21}^\beta. \end{aligned} \quad (41)$$

We have $\|e_{21}^\alpha\|, \|e_{21}^\beta\| \leq 1$ as b_{ij}, b_{ijk} are unit vectors. And the angular velocity error vector is defined as

$$e_{\Omega_2} = \Omega_2 - \Omega_2^d, \quad (42)$$

where Ω_2^d can be determined by (27). The properties of relative error variables are summarized as follows:

Proposition 2: (i) The error function Ψ_{21} and the error vector e_{21} can be rewritten as

$$\begin{aligned} \Psi_{21} &= \text{tr}[K_{21}(I_{3 \times 3} - R_1 Q_{21}^d R_2^T)], \\ e_{21} &= [Q_{21}^{dT} R_1^T K_{21} R_2 - R_2^T K_{21} R_1 Q_{21}^d]^V, \end{aligned}$$

where $K_{21} = k_{21}^\alpha s_{21} s_{21}^T + k_{21}^\beta s_{213} s_{213}^T \in \mathbb{R}^{3 \times 3}$.

(ii) Ψ_{21} is locally quadratic, more explicitly,

$$\underline{\psi}_{21} \|e_{21}\|^2 \leq \Psi_{21} \leq \bar{\psi}_{21} \|e_{21}\|^2, \quad (43)$$

where $\underline{\psi}_{21} = \frac{n_1}{n_2 + n_3}$ and $\bar{\psi}_{21} = \frac{n_1 n_4}{n_5(n_1 - \phi_{21})}$, for

$$\begin{aligned} n_1 &= 2 \min\{k_{21}^\alpha, k_{21}^\beta\}, \\ n_2 &= 4 \max\{(k_{21}^\alpha - k_{21}^\beta)^2, (k_{21}^\alpha)^2, (k_{21}^\beta)^2\}, \\ n_3 &= 4(k_{21}^\alpha + k_{21}^\beta)^2, \\ n_4 &= 2(k_{21}^\alpha + k_{21}^\beta), \\ n_5 &= 4 \min\{k_{21}^{\alpha 2}, k_{21}^{\beta 2}\}, \end{aligned}$$

and ϕ_{21} is a positive constant satisfying $\Psi_{21} < \phi_{21} < n_1$.

(iii) $\frac{d}{dt} \Psi_{21} = e_{21} \cdot e_{\Omega_2} + \Gamma_{21} \|e_{21}\|^2$.

(iv) $\|\frac{d}{dt} e_{21}\| = \frac{1}{\sqrt{2}}(k_{21}^\alpha + k_{21}^\beta) \|e_{\Omega_2}\| + (B_{\Omega_d} + B_{21}) \|e_{21}\|$.

Proof: Seen Appendix, Section C. ■

C. Position Tracking Error Variables

The position and velocity error vectors of Spacecraft 1 are given as

$$e_{x_1} = x_1 - x_1^d, \quad (44)$$

$$e_{v_1} = v_1 - v_1^d = \dot{e}_{x_1}. \quad (45)$$

Similarly, the relative position and relative velocity error vectors are given by

$$e_{x_{21}} = x_{21} - x_{21}^d, \quad (46)$$

$$e_{v_{21}} = v_{21} - v_{21}^d = \dot{e}_{x_{21}}, \quad (47)$$

IV. FORMATION CONTROL SYSTEM

Based on the error variables defined at the previous section, control systems are designed as follows.

A. Formation Control for Two Spacecraft

Proposition 3: Consider the formation of two spacecraft shown in Fig 1. For positive constants, k_{Ω_i} , k_{x_i} , and k_{v_i} , $i \in \{1, 2\}$, the control inputs are chosen as follows

$$u_1 = -e_b - k_{\Omega_1} e_{\Omega_1} + \hat{\Omega}_1^d J_1 (e_{\Omega_1} + \Omega_1^d) + J_1 \dot{\Omega}_1^d, \quad (48)$$

$$u_2 = -e_{21} - k_{\Omega_2} e_{\Omega_2} + \hat{\Omega}_2^d J_2 (e_{\Omega_2} + \Omega_2^d) + J_2 \dot{\Omega}_2^d, \quad (49)$$

$$f_1 = -k_{x_1} e_{x_1} - k_{v_1} e_{v_1} + m_1 \ddot{x}_1^d, \quad (50)$$

$$f_2 = -k_{x_2} e_{x_{21}} - k_{v_2} e_{v_{21}} + m_2 (\ddot{x}_1 + \ddot{x}_{21}^d). \quad (51)$$

Then, the zero equilibrium of tracking errors is exponentially stable.

Proof: The error dynamics are as follows. From (1), we have

$$\dot{e}_{v_1} = \frac{f_1}{m_1} - \ddot{x}_1^d, \quad \dot{e}_{v_{21}} = \frac{f_2}{m_2} - \ddot{x}_1 - \ddot{x}_{21}^d, \quad (52)$$

By using (3), (35), (42), we can obtain

$$J_i \dot{e}_{\Omega_i} = [J(e_{\Omega_i} + \Omega_i^d)]^\wedge e_{\Omega_i} - \hat{\Omega}_i^d J_i (e_{\Omega_i} + \Omega_i^d) - J_i \dot{\Omega}_i^d + u_i, \quad (53)$$

for $i = \{1, 2\}$.

For positive constants c_r and c_t , let the Lyapunov function be

$$\mathcal{V} = \mathcal{V}_1^r + \mathcal{V}_{21}^r + \mathcal{V}_1^t + \mathcal{V}_{21}^t, \quad (54)$$

where

$$\mathcal{V}_1^r = \frac{1}{2} e_{\Omega_1} \cdot J_1 e_{\Omega_1} + \Psi_1 + c_r J_1 e_{\Omega_1} \cdot e_b, \quad (55)$$

$$\mathcal{V}_{21}^r = \frac{1}{2} e_{\Omega_2} \cdot J_2 e_{\Omega_2} + \Psi_{21} + c_r J_2 e_{\Omega_2} \cdot e_{21}, \quad (56)$$

$$\mathcal{V}_1^t = \frac{1}{2} k_{x_1} e_{x_1}^\top e_{x_1} + \frac{1}{2} m_1 e_{v_1}^\top e_{v_1} + c_t e_{x_1}^\top e_{v_1}, \quad (57)$$

$$\mathcal{V}_{21}^t = \frac{1}{2} k_{x_2} e_{x_{21}}^\top e_{x_{21}} + \frac{1}{2} m_2 e_{v_{21}}^\top e_{v_{21}} + c_t e_{x_{21}}^\top e_{v_{21}}. \quad (58)$$

From (38), (43), the Lyapunov function is positive-definite about the equilibrium $(e_b, e_{21}, e_{\Omega_i}, e_{x_i}, e_{v_i}) = (0, 0, 0, 0, 0)$ for $i \in \{1, 2\}$, provided that the constants c_r, c_t are sufficiently small.

The time-derivative of $\mathcal{V}_1^r, \mathcal{V}_{21}^r$ are given by

$$\begin{aligned} \dot{\mathcal{V}}_1^r &= (e_{\Omega_1} + c_r e_b)^\top J_1 \dot{e}_{\Omega_1} + \dot{\Psi}_1 + c_r J_1 e_{\Omega_1}^\top \dot{e}_b, \\ \dot{\mathcal{V}}_{21}^r &= (e_{\Omega_2} + c_r e_{21})^\top J_2 \dot{e}_{\Omega_2} + \dot{\Psi}_{21} + c_r J_2 e_{\Omega_2}^\top \dot{e}_{21}. \end{aligned}$$

By using the properties (iv),(v) of Proposition 1, and the properties (iii),(iv) of Proposition 2, and also substituting

(53), (48), and (49), the time-derivative of $\mathcal{V}_1^r, \mathcal{V}_{21}^r$ can be rearranged as

$$\begin{aligned} \dot{\mathcal{V}}_1^r &\leq -[k_{\Omega_1} - c_r \lambda_{M_1} (\frac{1}{\sqrt{2}} + 1) \bar{k}_b] \|e_{\Omega_1}\|^2 \\ &\quad + c_r [\lambda_{M_1} (2B_{\Omega_d} + B_1) + k_{\Omega_1}] \|e_{\Omega_1}\| \|e_b\| \\ &\quad - (c_r - \Gamma) \|e_b\|^2, \end{aligned} \quad (59)$$

$$\begin{aligned} \dot{\mathcal{V}}_{21}^r &\leq -[k_{\Omega_2} - c_r \lambda_{M_2} (\frac{1}{\sqrt{2}} + 1) \bar{k}_{21}] \|e_{\Omega_2}\|^2 \\ &\quad + c_r [\lambda_{M_2} (2B_{\Omega_d} + B_{21}) + k_{\Omega_2}] \|e_{\Omega_2}\| \|e_{21}\| \\ &\quad - (c_r - \Gamma_{21}) \|e_{21}\|^2, \end{aligned} \quad (60)$$

For the translational dynamics, from (45) and (47), we have

$$\begin{aligned} \dot{\mathcal{V}}_1^t &= k_{x_1} e_{x_1}^\top e_{v_1} + c_t \|e_{v_1}\|^2 + (m_1 e_{v_1} + c e_{x_1})^\top \dot{e}_{v_1} \\ \dot{\mathcal{V}}_{21}^t &= k_{x_2} e_{x_{21}}^\top e_{v_{21}} + c_t \|e_{v_{21}}\|^2 + (m_2 e_{v_{21}} + c e_{x_{21}})^\top \dot{e}_{v_{21}}. \end{aligned}$$

Substituting (52) with the control inputs (50), (51), we obtain

$$\dot{\mathcal{V}}_1^t = -\frac{c k_{x_1}}{m_1} \|e_{x_1}\|^2 - (k_{v_1} - c_t) \|e_{v_1}\|^2 - \frac{c_t k_{v_1}}{m_1} e_{x_1}^\top e_{v_1}, \quad (61)$$

$$\dot{\mathcal{V}}_{21}^t = -\frac{c k_{x_2}}{m_2} \|e_{x_{21}}\|^2 - (k_{v_2} - c_t) \|e_{v_{21}}\|^2 - \frac{c_t k_{v_2}}{m_2} e_{x_{21}}^\top e_{v_{21}}, \quad (62)$$

From (59), (60), (61) and (62), the time-derivative of the complete Lyapunov function \mathcal{V} can be written as

$$\dot{\mathcal{V}} \leq -(\zeta_1^\top M_1 \zeta_1 + \zeta_{21}^\top M_{21} \zeta_{21} + \xi_1^\top N_1 \xi_1 + \xi_{21}^\top N_{21} \xi_{21}), \quad (63)$$

where $\zeta_1 = [\|e_b\|, \|e_{\Omega_1}\|]^\top$, $\zeta_{21} = [\|e_{21}\|, \|e_{\Omega_2}\|]^\top$, $\xi_1 = [\|e_{x_1}\|, \|e_{v_1}\|]^\top$ and $\xi_{21} = [\|e_{x_{21}}\|, \|e_{v_{21}}\|]^\top \in \mathbb{R}^2$, and the matrices $M_1, M_{21}, N_1, N_{21} \in \mathbb{R}^{2 \times 2}$ are defined as

$$\begin{aligned} M_1 &= \frac{1}{2} \begin{bmatrix} 2(c_r - \Gamma) & -c_r \Lambda_1 \\ -c_r \Lambda_1 & 2k_{\Omega_1} - c_r \lambda_{M_1} \bar{k}_b (\sqrt{2} + 2) \end{bmatrix}, \\ M_{21} &= \frac{1}{2} \begin{bmatrix} 2(c_r - \Gamma_{21}) & -c_r \Lambda_2 \\ -c_r \Lambda_2 & 2k_{\Omega_2} - c_r \lambda_{M_2} \bar{k}_{21} (\sqrt{2} + 2) \end{bmatrix}, \\ N_1 &= \frac{1}{2m_1} \begin{bmatrix} 2c_t k_{x_1} & c_t k_{v_1} \\ c_t k_{v_1} & 2m_1 (k_{v_1} - c_t) \end{bmatrix}, \\ N_{21} &= \frac{1}{2m_2} \begin{bmatrix} 2c_t k_{x_{21}} & c_t k_{v_{21}} \\ c_t k_{v_{21}} & 2m_2 (k_{v_2} - c_t) \end{bmatrix}, \end{aligned}$$

where $\Lambda_1 = \lambda_{M_1} (2B_{\Omega_d} + B_1) + k_{\Omega_1} \in \mathbb{R}$ and $\Lambda_2 = \lambda_{M_2} (2B_{\Omega_d} + B_{21}) + k_{\Omega_2} \in \mathbb{R}$. If the constants c_t and c_r are sufficiently small, we can show that all of matrices M_1, M_{21}, N_1 and N_{21} are positive definite. This implies that the equilibrium $(e_b, e_{21}, e_{\Omega_i}, e_{x_i}, e_{v_i}) = (0, 0, 0, 0, 0)$ for $i \in \{1, 2\}$, i.e., the desired formation, is exponentially stable. \blacksquare

B. Formation Control for Multiple Spacecraft

The preceding results for two spacecraft are readily generalized for an arbitrary number of spacecraft. Here, we present the controller structures as follows, without stability proof

that can be obtained by generalizing the proof of Proposition 3.

Proposition 4: Consider n spacecraft in the formation, the i -th spacecraft is paired serially with the $(i-1)$ -th spacecraft for $i \in \{2, 3, \dots, n\}$. For positive constants, k_{Ω_1} , k_{x_1} , k_{v_1} , k_{Ω_i} , k_{x_i} , and k_{v_i} , for $i \in \{2, 3, \dots, n\}$, the control inputs are chosen as follows

$$u_1 = -e_b - k_{\Omega_1} e_{\Omega_1} + \hat{\Omega}_1^d J_1 (e_{\Omega_1} + \Omega_1^d) + J_1 \dot{\Omega}_1^d, \quad (64)$$

$$u_i = -e_{i,i-1} - k_{\Omega_i} e_{\Omega_i} + \hat{\Omega}_i^d J_i (e_{\Omega_i} + \Omega_i^d) + J_i \dot{\Omega}_i^d, \quad (65)$$

$$f_1 = -k_{x_1} e_{x_1} - k_{v_1} e_{v_1} + m_1 \ddot{x}_1^d, \quad (66)$$

$$f_i = -k_{x_i} e_{x_{i,i-1}} - k_{v_i} e_{v_{i,i-1}} + m_i (\ddot{x}_{i-1} + \ddot{x}_{i,i-1}^d). \quad (67)$$

Then the zero equilibrium of tracking errors is exponentially stable.

V. NUMERICAL SIMULATION

Two simulation results are presented: (i) formation tracking control for two spacecraft, and (ii) formation stabilization for four spacecraft.

A. Formation Tracking for Two Spacecraft

Suppose $n = 2$. The mass and the inertia matrix are chosen as $m_1 = m_2 = 30$ kg and $J_1 = J_2 = \text{diag}[3, 2, 1]$ kgm². The desired absolute attitude of the leader, namely $R_1^d(t)$ is specified in terms of 3-2-1 Euler angles $(\alpha(t), \beta(t), \gamma(t))$, where

$$\alpha(t) = 0, \beta(t) = -0.7 + \cos(0.2t), \gamma(t) = 0.5 + \sin(2t).$$

The desired relative attitude $Q_{21}^d(t)$ is also defined in terms of another set of Euler angles given by

$$\phi(t) = \sin(0.5t), \theta(t) = 2, \psi(t) = \cos(t) + 1.$$

The initial attitudes for Spacecraft 1 and Spacecraft 2 are chosen as $R_1(0) = R_2(0) = I_{3 \times 3}$. The initial angular velocity is chosen to be zero for both spacecraft.

For the translational motion, the desired position vectors are given by

$$x_1^d = [\sin(0.04t), 0, -\sin(0.07t)]^T \\ x_{21}^d = [2, -3 + \cos(0.02t), 10]^T.$$

The initial positions are chosen as $x_1 = [0, 0, 0]$ and $x_2 = [2, -1, 7]$. Control gains are selected to be $k_{\Omega_1} = k_{\Omega_2} = 7$, $k_{\Omega_1}^d = k_{b_1} = 25$, $k_{\Omega_2}^d = k_{b_2} = 25.1$, $k_{x_1} = k_{x_2} = 49$, and $k_{v_1} = k_{v_2} = 12.6$.

The corresponding numerical results are illustrated in Fig 2, where the attitude error vectors are defined as

$$e_{R_1} = \frac{1}{2}(R_1^{d\top} R_1 - R_1^\top R_1^d)^\vee, \\ e_{Q_{21}} = \frac{1}{2}(Q_{21}^{d\top} Q_{21} - Q_{21}^\top Q_{21}^d)^\vee.$$

It is illustrated that tracking errors are nicely converted to zero.

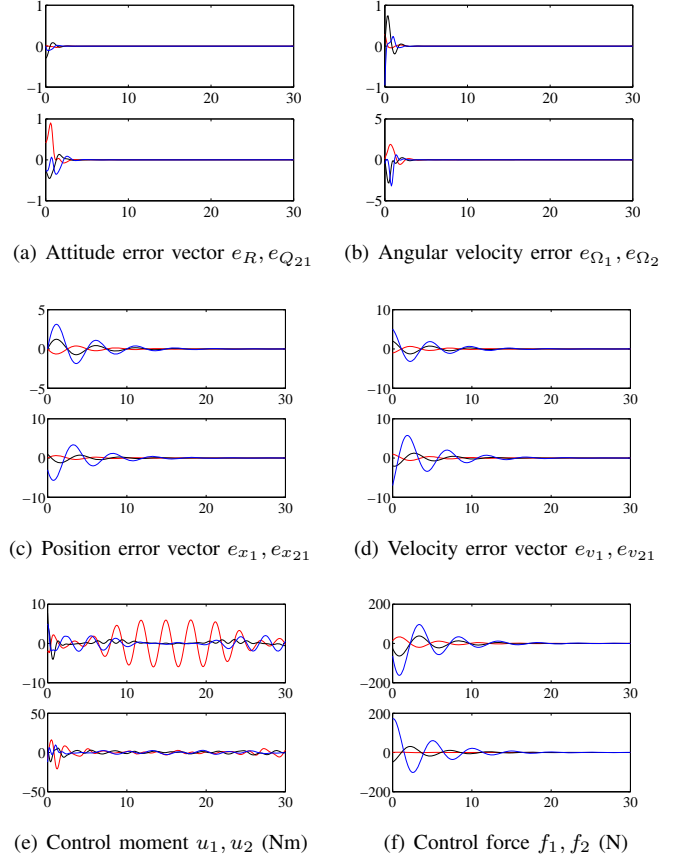


Fig. 2. Numerical results for two spacecraft in formation.

B. Formation Control for Four Spacecraft

Next, we consider formation control for $n = 4$ spacecraft. The mass and the inertia matrix are same as the previous case. The desired attitudes are chosen as $R_1^d = Q_{12}^d = Q_{23}^d = Q_{34}^d = I_{3 \times 3}$. This represents attitude synchronization. The initial attitudes are given as

$$R_1(0) = \exp(0.2\pi\hat{e}_2), \quad R_2(0) = \exp(0.5\pi\hat{e}_1), \\ R_3(0) = \exp(0.4\pi\hat{e}_1), \quad R_4(0) = \exp(0.8\pi\hat{e}_3),$$

where $e_1 = [1, 0, 0]^T$, $e_2 = [0, 1, 0]^T$, and $e_3 = [0, 0, 1]^T$. Also, the initial angular velocity is chosen to be zero for each spacecraft.

The desired position trajectories are chosen as

$$x_1^d = [-100, 0, 0]^T, \\ x_{21}^d = [100, 100, 6]^T, \quad x_{32}^d = x_{43}^d = [0, -200, 0]^T.$$

The initial conditions are $x_1 = [-200, 0, 0]^T$, $x_2 = [-100, -50, 0]^T$, $x_3 = [0, 0, 20]^T$, $x_4 = [100, 100, -1]^T$, $v_1 = [0, 0, 0]^T$, $v_2 = [0, 0, 10]^T$ and $v_3 = v_4 = [0, 10, 0]^T$.

The corresponding numerical results is illustrated by Figure 4, where the attitude error vectors are defined as

$$e_{R_1} = \frac{1}{2}(R_1^{d\top} R_1 - R_1^\top R_1^d)^\vee \in \mathbb{R}^3, \\ e_{Q_{i,i-1}} = \frac{1}{2}(Q_{i,i-1}^{d\top} Q_{i,i-1} - Q_{i,i-1}^\top Q_{i,i-1}^d)^\vee \in \mathbb{R}^3.$$

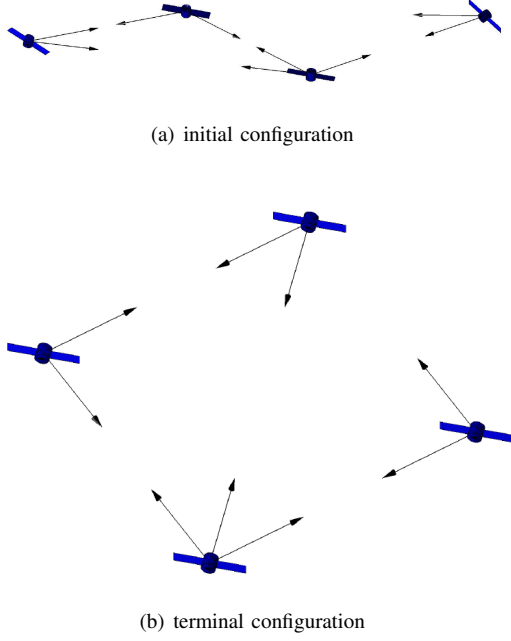


Fig. 3. The initial configuration and the terminal configuration of four spacecraft

The initial configuration and the terminal configuration of spacecraft are also illustrated at Figure 3.

APPENDIX

A. Lemmas

Lemma 1: The properties stated in Proposition 1 of [15] are summarized and extended as follows: For non-negative constants f_1, f_2, f_3 , let $F = \text{diag}[f_1, f_2, f_3] \in \mathbb{R}^{3 \times 3}$, and let $P \in \text{SO}(3)$. Define

$$\Phi = \frac{1}{2} \text{tr}[F(I_{3 \times 3} - P)], \quad (68)$$

$$e_P = \frac{1}{2}(FP - P^T F)^\vee. \quad (69)$$

Then, Φ is bounded by the square of the norm of e_P as

$$\frac{h_1}{h_2 + h_3} \|e_P\|^2 \leq \Phi \leq \frac{h_1 h_4}{h_5(h_1 - \phi)} \|e_P\|^2. \quad (70)$$

If $\Phi < \phi < h_1$ for a constant ϕ , where h_i are given by

$$\begin{aligned} h_1 &= \min\{f_1 + f_2, f_2 + f_3, f_3 + f_1\}, \\ h_2 &= \max\{(f_1 - f_2)^2, (f_2 - f_3)^2, (f_3 - f_1)^2\}, \\ h_3 &= \max\{(f_1 + f_2)^2, (f_2 + f_3)^2, (f_3 + f_1)^2\}, \\ h_4 &= \max\{f_1 + f_2, f_2 + f_3, f_3 + f_1\}, \\ h_5 &= \min\{(f_1 + f_2)^2, (f_2 + f_3)^2, (f_3 + f_1)^2\}. \end{aligned}$$

Proof: From Rodrigues' formula, we have $P = \exp(\hat{y})$, for $y \in \mathbb{R}^3$. According to [15], using the MATLAB symbolic computation tool, we find

$$\|e_P\|^2 = \frac{(1 - \cos \|y\|)^2}{4\|y\|^4} \sum_{(i,j,k) \in \mathcal{C}} (f_i - f_j)^2 y_i^2 y_j^2$$

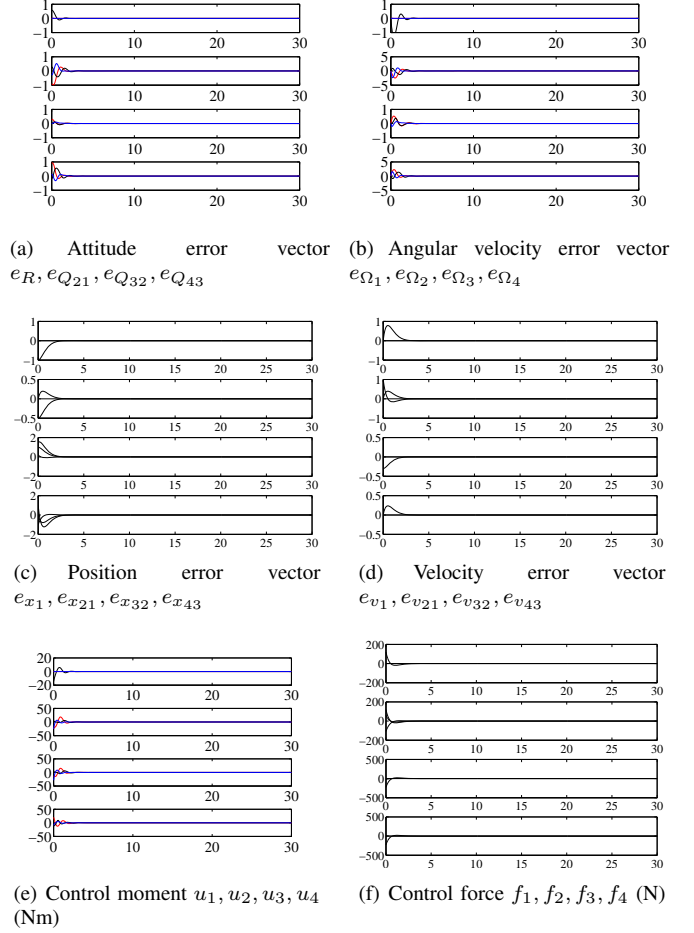


Fig. 4. Numerical results for four spacecraft in formation.

$$+ \frac{\sin^2 \|y\|}{4\|y\|^2} \sum_{(i,j,k) \in \mathcal{C}} (f_i + f_j)^2 y_k^2, \quad (71)$$

$$\Phi = \frac{1 - \cos \|y\|}{2\|y\|^2} \sum_{(i,j,k) \in \mathcal{C}} (f_i + f_j) y_k^2. \quad (72)$$

Using these two equations it follows that

$$\frac{\|e_P\|^2}{\Phi} \leq \frac{h_2 + h_3}{h_1}, \quad \frac{\Phi}{\|e_P\|^2} \leq \frac{h_1 h_4}{h_5(h_1 - \phi)},$$

which shows (70). Lemma 1 is purely the citation from [15]. \blacksquare

Lemma 2: Consider

$$\Phi' = \frac{1}{2} \text{tr}[F'(I_{3 \times 3} - P')], \quad (73)$$

$$e_{P'} = \frac{1}{2}(F'P' - P'^T F')^\vee, \quad (74)$$

where $F' = \text{diag}[f'_1, f'_2, f'_3] \in \mathbb{R}^{3 \times 3}$, for positive scalars f'_i , $i = \{1, 2, 3\}$, and $P' = A^T P A$, for any rotation matrix $A \in \text{SO}(3)$. We say that Φ' and $e_{P'}$ are bounded respectively in the form of

$$(i) \quad \frac{h_1}{h'_2 + h'_3} \|e_{P'}\|^2 \leq \Phi' \leq \frac{h_1 h'_4}{h'_5(h_1 - \phi)} \|e_{P'}\|^2,$$

$$(ii) \quad \|e_{P'}\| \leq \sqrt{\frac{2h'_2 + h'_5}{h'_5}} \|e_{P'}\|,$$

where h_i are given by

$$\begin{aligned} h'_2 &= \max\{(f'_1 - f'_2)^2, (f'_2 - f'_3)^2, (f'_3 - f'_1)^2\}, \\ h'_3 &= \max\{(f'_1 + f'_2)^2, (f'_2 + f'_3)^2, (f'_3 + f'_1)^2\}, \\ h'_4 &= \max\{f'_1 + f'_2, f'_2 + f'_3, f'_3 + f'_1\}, \\ h'_5 &= \min\{(f'_1 + f'_2)^2, (f'_2 + f'_3)^2, (f'_3 + f'_1)^2\}. \end{aligned}$$

Proof: We start from finding the boundedness of Φ' , employing (9) leads to

$$P' = A^\top P A = A^\top \exp(\hat{y}) A = \exp(\widehat{A^\top y}).$$

From Rodrigues' formula, we know $P' = \exp(\hat{y}')$ for $y' \in \mathbb{R}^3$. Evidently, we have

$$y' = A^\top y, \quad \|y'\| = \|y\|.$$

We therefore write

$$\begin{aligned} \Phi' &= \frac{(1 - \cos \|y'\|)}{2\|y'\|^2} \sum_{(i,j,k) \in \mathfrak{C}} (f'_i + f'_j) y_k'^2 \\ &= \frac{(1 - \cos \|y\|)}{2\|y\|^2} \sum_{(i,j,k) \in \mathfrak{C}} (f'_i + f'_j) y_k^2. \end{aligned} \quad (75)$$

Comparing (71) with (75) yields

$$\frac{\|e_P\|^2}{\Phi} \leq \frac{h'_2 + h'_3}{h_1}, \quad \frac{\Phi}{\|e_P\|^2} \leq \frac{h_1 h'_4}{h_5(h_1 - \phi)},$$

which shows (i).

Similar to (71), we can have

$$\begin{aligned} \|e_{P'}\|^2 &= \frac{(1 - \cos \|x'\|)^2}{4\|x'\|^4} \sum_{(i,j,k) \in \mathfrak{C}} (f'_i - f'_j)^2 x_i'^2 x_j'^2 \\ &\quad + \frac{\sin^2 \|x'\|}{4\|x'\|^2} \sum_{(i,j,k) \in \mathfrak{C}} (f'_i + f'_j)^2 x_k'^2. \end{aligned}$$

By comparing the magnitude e_P and $e'_{P'}$

$$\begin{aligned} \frac{\|e_{P'}\|^2}{\|e_P\|^2} &\leq \frac{1 + \cos \|x\|}{\|x\|^2} \frac{\max\{(\bar{f}_i - \bar{f}_j)^2\} \bar{x}_i^2 \bar{x}_j^2}{\min\{(f_i + f_j)^2\} x_k^2} \\ &\quad + \frac{\max\{(\bar{f}_i + \bar{f}_j)^2\} \|\bar{x}\|^2}{\min\{(f_i + f_j)^2\} \|x\|^2} = (1 + \cos \|x\|) \frac{\bar{h}_2}{h_5} + \frac{\bar{h}_5}{h_5} \\ &\leq \frac{2\bar{h}_2 + \bar{h}_5}{h_5}, \end{aligned}$$

which leads to (ii). It shows that $P' = A^\top P A$, for any rotation matrix $A \in \text{SO}(3)$, we can still bound the new Φ' and $e'_{P'}$ in terms of original $\|e_P\|$. ■

B. Proof of Proposition 1

Property (i), (ii) has been verified in [14]. Also, it has been shown that

$$K_1 = U_1 G_1 U_1^\top, \quad (76)$$

where $U_1 \in \text{SO}(3)$ and $G_1 = \text{diag}[k_{b_A}, k_{b_B}, 0]$ such that

$$\text{tr}[K_1] = \text{tr}[G_1] = k_{b_A} + k_{b_B} \triangleq \bar{k}_b. \quad (77)$$

From (33), we can tell that the magnitude of e_{b_i} is bounded, since b_i and b_{i_d} are unit vectors, for $i = \{A, B\}$. That is, $\|e_{b_i}\| \leq 1$, which leads to

$$\|e_b\| \leq k_{b_A} \|e_{b_A}\| + k_{b_B} \|e_{b_B}\| \leq k_{b_A} + k_{b_B} = \bar{k}_b. \quad (78)$$

which shows (iii).

From (36), Differentiating Ψ_1 gives us

$$\dot{\Psi}_1 = -\text{tr}[K_1 \dot{R}_1^e] + \text{tr}[\dot{K}_1 (I_{3 \times 3} - R_1^e)]. \quad (79)$$

From (4) and (23), the first term at right hand side can be written as

$$-\text{tr}[K_1 \dot{R}_1^e] = -\text{tr}[\hat{\Omega}_1 R_1^{d\top} K R_1] + \text{tr}[\hat{\Omega}_1^d R_1^{d\top} K R_1].$$

Applying one of the hat map properties (6) leads us to

$$\begin{aligned} -\text{tr}[K_1 \dot{R}_1^e] &= \Omega_1^\top (R_1^{d\top} K_1 R_1 - R_1^\top K_1 R_1^d)^\vee \\ &\quad - \Omega_1^{d\top} (R_1^{d\top} K_1 R_1 - R_1^\top K_1 R_1^d)^\vee \\ &= (\Omega_1 - \Omega_1^d)^\top e_b = e_{\Omega_1}^\top e_b. \end{aligned} \quad (80)$$

Next, we can write

$$\begin{aligned} \text{tr}[\dot{K}_1 (I_{3 \times 3} - R_1^e)] &= \text{tr}[\tilde{U}_1 \tilde{G}_1 \tilde{U}_1^\top (I - R_1^e)] \\ &= \text{tr}[\tilde{G}_1] - \text{tr}[\tilde{G}_1 \tilde{U}_1^\top R_1^e \tilde{U}_1] \\ &= \text{tr}[\tilde{G}_1 (I_{3 \times 3} - \tilde{U}_1^\top R_1^e \tilde{U}_1)] \triangleq \tilde{\Phi}_1, \end{aligned} \quad (81)$$

where $\tilde{K}_1 \triangleq \tilde{U}_1 \tilde{G}_1 \tilde{U}_1^\top$ is symmetric. We have $\tilde{U}_1 \in \text{SO}(3)$ and $\tilde{G}_1 = \text{diag}[\tilde{g}_1, \tilde{g}_2, \tilde{g}_3] \in \mathbb{R}^{3 \times 3}$ where \tilde{g}_i are positive constants. Then, Property (ii) of Lemma 2 is applied here. That is, we let $F' = 2\tilde{G}_1$ and $P' = \tilde{U}_1^\top R_1^e \tilde{U}_1$ to obtain

$$\frac{h_1}{\tilde{h}_2 + \tilde{h}_3} \|e_b\|^2 \leq \tilde{\Phi}_1 \leq \frac{h_1 \tilde{h}_4}{h_5(h_1 - \tilde{\phi}_1)} \|e_b\|^2. \quad (82)$$

If $\tilde{\Phi}_1 < \tilde{\phi}_1 < \tilde{h}_1$ for a constant ϕ_1 , where \tilde{h}_i are given by

$$\begin{aligned} \tilde{h}_2 &= 4\max\{(\tilde{g}_1 - \tilde{g}_2)^2, (\tilde{g}_2 - \tilde{g}_3)^2, (\tilde{g}_3 - \tilde{g}_1)^2\}, \\ \tilde{h}_3 &= 4\max\{(\tilde{g}_1 + \tilde{g}_2)^2, (\tilde{g}_2 + \tilde{g}_3)^2, (\tilde{g}_3 + \tilde{g}_1)^2\}, \\ \tilde{h}_4 &= 2\max\{\tilde{g}_1 + \tilde{g}_2, \tilde{g}_2 + \tilde{g}_3, \tilde{g}_3 + \tilde{g}_1\}. \end{aligned}$$

Summation of (80) and (82) allows us to write

$$\dot{\Psi}_1 \leq e_{\Omega_1}^\top e_b + \Gamma_1 \|e_b\|,$$

where $\Gamma_1 = \frac{h_1 \tilde{h}_4}{h_5(h_1 - \phi_1)}$. This shows (iv).

Differentiating (37), we are able to write

$$\begin{aligned} \dot{e}_b &= \dot{e}_b^\zeta + \dot{e}_b^\xi, \\ \dot{e}_b^\zeta &= \dot{R}_1^{d\top} K_1 R_1 + R_1^{d\top} K_1 \dot{R}_1 - \dot{R}_1^\top K_1 R_1^d - R_1^\top K_1 \dot{R}_1^d, \\ \dot{e}_b^\xi &= R_1^{d\top} \dot{K}_1 R_1 - R_1^\top \dot{K}_1 R_1^d. \end{aligned}$$

Using kinematics equations of the spacecraft, we know that

$$\begin{aligned} \dot{e}_b^\zeta &= (\hat{\Omega}_1 R_1^\top K_1 R_1^d + R_1^{d\top} K_1 R_1 \hat{\Omega}_1) \\ &\quad - (\hat{\Omega}_1^d R_1^{d\top} K_1 R_1 + R_1^\top K_1 R_1^d \hat{\Omega}_1^d). \end{aligned}$$

Inserting $\Omega_1 = e_{\Omega_1} + \Omega_1^d$ and rearrangement lead us to

$$\dot{e}_b^\zeta = (\hat{e}_{\Omega_1} R_1^\top K_1 R_1^d + R_1^{d\top} K_1 R_1 \hat{e}_{\Omega_1}) + \hat{e}_b \hat{\Omega}_1^d - \hat{\Omega}_1^d \hat{e}_b. \quad (83)$$

Further, in view of (7) and (5), we obtain

$$e_b^\zeta = E_{\Omega_1} e_{\Omega_1} + e_b \times \Omega_1^d, \quad (84)$$

where $E_{\Omega_1} = \{\text{tr}[R_1^\top K_1 R_1^d] I_{3 \times 3} - R_1^\top K_1 R_1^d\} \in \mathbb{R}^{3 \times 3}$. In particular, the Frobenius norm of E_{Ω_1} is given by

$$\begin{aligned} \|E_{\Omega_1}\|_F &= \sqrt{\text{tr}[E_{\Omega_1}^\top E_{\Omega_1}]} = \sqrt{\text{tr}[R_1^\top K_1 R_1^d]^2 - \text{tr}[K_1^2]} \\ &\leq \frac{1}{\sqrt{2}} \text{tr}[G_1] = \frac{1}{\sqrt{2}} \bar{k}_b, \end{aligned}$$

where (77) is applied. From (29), we can write

$$\|e_b^\zeta\| \leq \frac{1}{\sqrt{2}} \bar{k}_b \|e_{\Omega_1}\| + B_{\Omega_d} \|e_b\|. \quad (85)$$

As for \hat{e}_b^ξ , we know that

$$\hat{e}_b^\xi = R_1^{d\top} (\tilde{U}_1 \tilde{G}_1 \tilde{U}_1^\top) R_1 - R_1^\top (\tilde{U}_1 \tilde{G}_1 \tilde{U}_1^\top) R_1^d,$$

and this yields

$$\begin{aligned} \|\hat{e}_b^\xi\| &= \|\tilde{U}_1^\top R_1^d (\hat{e}_b^\xi) R_1^\top \tilde{U}_1\| \\ &= \|\tilde{G}_1 \tilde{U}_1^\top R_1 R_1^{d\top} \tilde{U}_1 - \tilde{U}_1^\top R_1^d R_1^\top \tilde{U}_1 \tilde{G}_1\| \\ &\triangleq \|\tilde{G}_1 \tilde{U}_1 - \tilde{U}_1^\top \tilde{G}_1\|, \end{aligned}$$

where $\tilde{U}_1 = \tilde{U}_1^\top R_1 R_1^{d\top} \tilde{U}_1 = \tilde{U}_1^\top R_1^e \tilde{U}_1$. Notice that we have applied $\|A\| = \|RA\| = \|AR\|$, for any matrix $A \in \mathbb{R}^{3 \times 3}$ and $R \in \text{SO}(3)$. Also we know the vee map does not change the magnitude of the vector, which implies

$$\|\tilde{G}_1 \tilde{U}_1 - \tilde{U}_1^\top \tilde{G}_1\| = \|(\tilde{G}_1 \tilde{U}_1 - \tilde{U}_1^\top \tilde{G}_1)^\vee\| = \|\hat{e}_b^\xi\|.$$

Using property (ii) of Lemma 2, we know that

$$\|\hat{e}_b^\xi\| \leq B_1 \|e_b\|, \quad (86)$$

where $B_1 = 2\sqrt{\frac{2\bar{h}_2 + \bar{h}_5}{h_5}}$. The summation of (85) and (86) results in

$$\|\hat{e}_b\| \leq \frac{1}{\sqrt{2}} \bar{k}_b \|e_{\Omega_1}\| + (B_{\Omega_d} + B_1) \|e_b\|, \quad (87)$$

which shows (v).

C. Proof of Proposition 2

The configuration error function (40) can be written as

$$\begin{aligned} \Psi_{21} &= k_{21}^\alpha [1 + b_{12} \cdot (Q_{21}^d b_{21})] + k_{21}^\beta [1 + b_{123} \cdot (Q_{21}^d b_{213})] \\ &= k_{21}^\alpha + k_{21}^\beta + \text{tr}[k_{21}^\alpha Q_{21}^d b_{21} b_{12}^\top + k_{21}^\beta Q_{21}^d b_{213} b_{123}^\top] \\ &= k_{21}^\alpha + k_{21}^\beta + \text{tr}[k_{21}^\alpha Q_{21}^d (R_2^\top s_{12}) (s_{12}^\top R_1) \\ &\quad + k_{21}^\beta Q_{21}^d (R_2^\top s_{213}) (s_{123}^\top R_1)], \end{aligned}$$

employing the geometric constraints $s_{12} = -s_{21}$ and $s_{123} = -s_{213}$, we obtain

$$\begin{aligned} \Psi_{21} &= k_{21}^\alpha + k_{21}^\beta - \text{tr}[Q_{21}^d R_2^\top (k_{21}^\alpha s_{21} s_{21}^\top + k_{21}^\beta s_{213} s_{213}^\top) R_1] \\ &\triangleq k_{21}^\alpha + k_{21}^\beta - \text{tr}[Q_{21}^d R_2^\top K_{21} R_1], \end{aligned} \quad (88)$$

where we define a symmetric matrix,

$$K_{21} = k_{21}^\alpha s_{21} s_{21}^\top + k_{21}^\beta s_{213} s_{213}^\top. \quad (89)$$

According to the spectral theory, the matrix K_{12} can be decomposed into $K_{12} = U_{12} G_{12} U_{12}^\top$, where G_{12} is the diagonal matrix given by $G_{12} = \text{diag}[k_{12}^\alpha, k_{12}^\beta, 0] \in \mathbb{R}^3$, and U_{12} is an orthonormal matrix defined as $U_{12} = [s_{12}, s_{123}, \frac{s_{12} \times s_{123}}{\|s_{12} \times s_{123}\|}] \in \text{SO}(3)$. Hence, the substitution of $k_{12}^\alpha + k_{12}^\beta = \text{tr}[G_{12}] = \text{tr}[K_{12}]$ leads us to

$$\Psi_{21} = \text{tr}[K_{21} (I_{3 \times 3} - R_1 Q_{21}^d R_2^\top)] \quad (90)$$

$$= \text{tr}[G_{21} (I_{3 \times 3} - U_{21}^\top R_1 Q_{21}^d R_2^\top U_{21})], \quad (91)$$

These are alternative expressions of Ψ_{21} . In addition, (41) can be written as

$$\begin{aligned} \hat{e}_{21} &= k_{21}^\alpha [(Q_{21}^d{}^\top b_{12}) \times b_{21}]^\wedge + k_{21}^\beta [(Q_{21}^d{}^\top b_{123}) \times b_{213}]^\wedge \\ &= k_{21}^\alpha [b_{21} b_{12}^\top Q_{21}^d - Q_{21}^d{}^\top b_{12} b_{21}^\top] \\ &\quad + k_{21}^\beta [b_{213} b_{123}^\top Q_{21}^d - Q_{21}^d{}^\top b_{123} b_{213}^\top] \\ &= k_{21}^\alpha [R_2^\top s_{21} s_{12}^\top R_1 Q_{21}^d - Q_{21}^d{}^\top R_1^\top s_{12} s_{21}^\top R_2] \\ &\quad + k_{21}^\beta [R_2^\top s_{213} s_{123}^\top R_1 Q_{21}^d - Q_{21}^d{}^\top R_1^\top s_{123} s_{213}^\top R_2] \\ &= Q_{21}^d{}^\top R_1^\top K_{21} R_2 - R_2^\top K_{21} R_1 Q_{21}^d, \end{aligned} \quad (92)$$

where one of the hat properties (5) is applied. Together with (90), property (i) is proved.

We apply Lemma 1 here, that is, in view of (91) and (68), let $F = 2G_{12}$ and $P = U_{21}^\top R_1 Q_{21}^d R_2^\top U_{21}$, we obtain $\Psi_{12} = \Phi$ and $\psi = \phi$, which results in

$$\frac{h_1}{h_2 + h_3} \|e_P\|^2 \leq \Psi_{12} \leq \frac{h_1 h_4}{h_5 (h_1 - \phi)} \|e_P\|^2. \quad (93)$$

Furthermore, substituting the new expression of F and P into (69), we obtain

$$\begin{aligned} \hat{e}_P &= G_{21} U_{21}^\top R_1 Q_{21}^d R_2^\top U_{21} - U_{21}^\top R_2 Q_{21}^d{}^\top R_1^\top U_{21} G_{21} \\ &= U_{21}^\top (K_{21} R_1 Q_{21}^d R_2^\top - R_1 Q_{21}^d{}^\top R_1^\top K_{21}) U_{21} \\ &= U_{21}^\top R_2 Q_{21}^d{}^\top (\hat{e}_{21}) Q_{21}^d R_2^\top U_{21}. \end{aligned}$$

Employing (8) yields to

$$\hat{e}_P = (U_{21}^\top R_2 Q_{21}^d{}^\top \hat{e}_{21})^\wedge.$$

This implies $\|e_P\| = \|\hat{e}_{21}\|$ since U_{21} , R_2 , Q_{21}^d all belongs to $\text{SO}(3)$. This shows (ii).

The time-derivative of (90) is given by

$$\dot{\Psi}_{21} = \text{tr}[\dot{K}_{21} (I_{3 \times 3} - Q_{21}^e)] - \text{tr}[K_{21} \dot{Q}_{21}^e], \quad (94)$$

where $Q_{21}^e = R_1 Q_{21}^d R_2^\top \in \text{SO}(3)$. Notice that \dot{K}_{12} is symmetric, since

$$\begin{aligned} \dot{K}_{21} &= k_{21}^\alpha \hat{\mu}_{21} s_{21} s_{21}^\top - k_{21}^\alpha s_{21} s_{21}^\top \hat{\mu}_{21} \\ &\quad + k_{21}^\beta \hat{\mu}_{213} s_{213} s_{213}^\top - k_{21}^\beta s_{213} s_{213}^\top \hat{\mu}_{213} = \dot{K}_{21}^\top, \end{aligned}$$

and it can be decomposed to $\dot{K}_{21} = \tilde{U}_{21} \tilde{G}_{21} \tilde{U}_{21}^\top$ where $\tilde{U}_{21} \in \text{SO}(3)$ and $\tilde{G}_{21} = \text{diag}[\tilde{g}_1, \tilde{g}_2, \tilde{g}_3]$ for three positive scalars. Hence, the first term on the right of (94) can be written as

$$\text{tr}[\dot{K}_{21} (I_{3 \times 3} - Q_{21}^e)] = \text{tr}[\tilde{G}_{21} (I_{3 \times 3} - \tilde{U}_{21}^\top Q_{21}^e \tilde{U}_{21})] \triangleq \tilde{\Phi}_{21}.$$

Here we apply the property (ii) of Lemma. That is, we let $F' = 2\tilde{G}_{21}$ and $P' = \tilde{U}_{21}^\top Q_{21}^e \tilde{U}_{21}$ to obtain

$$\frac{h_1}{\tilde{h}_2 + \tilde{h}_3} \|e_{21}\|^2 \leq \tilde{\Phi}_{21} \leq \frac{h_1 \tilde{h}_4}{h_5(h_1 - \psi)} \|e_{21}\|^2. \quad (95)$$

If $\tilde{\Phi} < \tilde{\phi} < \tilde{h}_1$ for a constant ϕ , where \tilde{h}_i are given by

$$\begin{aligned} \tilde{h}_2 &= 4\max\{(\tilde{g}_1 - \tilde{g}_2)^2, (\tilde{g}_2 - \tilde{g}_3)^2, (\tilde{g}_3 - \tilde{g}_1)^2\}, \\ \tilde{h}_3 &= 4\max\{(\tilde{g}_1 + \tilde{g}_2)^2, (\tilde{g}_2 + \tilde{g}_3)^2, (\tilde{g}_3 + \tilde{g}_1)^2\}, \\ \tilde{h}_4 &= 2\max\{\tilde{g}_1 + \tilde{g}_2, \tilde{g}_2 + \tilde{g}_3, \tilde{g}_3 + \tilde{g}_1\}. \end{aligned}$$

In addition, to find out the second term on the right of (94), we first find out the time derivative of Q_{12}^e ,

$$\begin{aligned} \dot{Q}_{21}^e &= \frac{d}{dt}(R_1 Q_{21}^d R_2^\top) \\ &= (R_1 \hat{\Omega}_1) Q_{21}^d R_2^\top + R_1 (Q_{21}^d \hat{\Omega}_2^d - \hat{\Omega}_1 Q_{21}^d) R_2^\top \\ &\quad + R_1 Q_{21}^d (-\hat{\Omega}_2 R_2^\top) \\ &= -R_1 Q_{21}^d (\hat{\Omega}_2 - \hat{\Omega}_2^d) R_2^\top = -R_1 Q_{21}^d (\hat{e}_{\Omega_2}) R_2^\top, \end{aligned}$$

where (27) and (8) are applied. Then we can write

$$\begin{aligned} -\text{tr}[K_{21} \dot{Q}_{21}^e] &= \text{tr}[K_{21} R_1 Q_{21}^d (\hat{e}_{\Omega_2}) R_2^\top] \\ &= \text{tr}[(\hat{e}_{\Omega_2}) R_2^\top K_{21} R_1 Q_{21}^d] \\ &= e_{\Omega_2}^\top (Q_{21}^d{}^\top R_1^\top K_{21} R_2 - R_2^\top K_{12} R_1 Q_{21}^d)^\vee \\ &= e_{\Omega_2}^\top e_{21}, \end{aligned} \quad (96)$$

where (6) and (92) are applied in the process of rearrangement. Next, substituting (95) and (96) to (94) results in

$$\dot{\Psi}_{12} \leq e_{\Omega_2}^\top e_{21} + \Gamma_{21} \|e_{21}\|^2, \quad (97)$$

where $\Gamma_{21} \triangleq \sqrt{\frac{h_1 \tilde{h}_4}{h_5(h_1 - \phi)}}$. This is the proof of (iii).

Lastly we show (iv). Differentiating (92) with respect to time, we obtain

$$\dot{\hat{e}}_{21} = \hat{e}_a + \hat{e}_b, \quad (98)$$

where

$$\begin{aligned} \hat{e}_a &= \dot{Q}_{21}^d{}^\top R_1^\top K_{21} R_2 + Q_{21}^d{}^\top \dot{R}_1^\top K_{21} R_2 + Q_{21}^d{}^\top R_1^\top K_{21} \dot{R}_2 \\ &\quad - \dot{R}_2^\top K_{21} R_1 Q_{21}^d - R_2^\top K_{21} \dot{R}_1 Q_{21}^d - R_2^\top K_{21} R_1 \dot{Q}_{21}^d, \\ \hat{e}_b &= Q_{21}^d{}^\top R_1^\top \dot{K}_{21} R_2 - R_2^\top \dot{K}_{21} R_1 Q_{21}^d. \end{aligned}$$

Employing the kinematic equations (4), (27) and one of the hat map properties (8) into \hat{e}_a leads to

$$\begin{aligned} \hat{e}_a &= -(\hat{\Omega}_2^d Q_{21}^d{}^\top R_1^\top K_{21} R_2 + R_2^\top K_{21} R_1 Q_{21}^d \hat{\Omega}_2^d) \\ &\quad + (\hat{\Omega}_2 R_2^\top K_{21} R_1 Q_{21}^d + Q_{21}^d{}^\top R_1^\top K_{21} R_2 \hat{\Omega}_2), \end{aligned}$$

and the substitution of $e_{\Omega_2} = \Omega_2 - \Omega_2^d$ allows us to write

$$\begin{aligned} \hat{e}_a &= (Q_{21}^d{}^\top R_1^\top K_{21} R_2 - R_2^\top K_{21} R_1 Q_{21}^d) \hat{\Omega}_2^d \\ &\quad - \hat{\Omega}_2^d (Q_{21}^d{}^\top R_1^\top K_{21} R_2 - R_2^\top K_{21} R_1 Q_{21}^d) \\ &\quad + (\hat{e}_{\Omega_2} R_2^\top K_{21} R_1 Q_{21}^d + Q_{21}^d{}^\top R_1^\top K_{21} R_2 \hat{e}_{\Omega_2}) \\ &= \hat{e}_{21} \hat{\Omega}_2^d - \hat{\Omega}_2^d \hat{e}_{21} \end{aligned}$$

$$+ (\hat{e}_{\Omega_2} R_2^\top K_{21} R_1 Q_{21}^d + Q_{21}^d{}^\top R_1^\top K_{21} R_2 \hat{e}_{\Omega_2}).$$

Apply (5) and (7) here. Then removing the hat map on both side of the equation, we obtain

$$\epsilon_a = e_{21} \times \Omega_2^d + \quad (99)$$

$$\begin{aligned} &\{\text{tr}[R_2^\top K_{21} R_1 Q_{21}^d] I_{3 \times 3} - R_2^\top K_{21} R_1 Q_{21}^d\} e_{\Omega_2} \\ &\triangleq e_{21} \times \Omega_2^d + E_{\Omega_2} e_{\Omega_2}. \end{aligned} \quad (100)$$

where $E_{\Omega_2} = \{\text{tr}[R_2^\top K_{21} R_1 Q_{21}^d] I_{3 \times 3} - R_2^\top K_{21} R_1 Q_{21}^d\}$. Moreover, the Frobenius norm of E_{Ω_2} is given by

$$\begin{aligned} \|E_{\Omega_2}\|_F &= \sqrt{\text{tr}[E_{\Omega_2}^\top E_{\Omega_2}]} \\ &= \frac{1}{2} \sqrt{\text{tr}[R_2^\top K_{21} R_1 Q_{21}^d]^2 + \text{tr}[K_{21}^2]}. \end{aligned} \quad (101)$$

Using the fact that $K_{21} = U_{21} G_{21} U_{21}^\top$, we find

$$\begin{aligned} \text{tr}[R_2^\top K_{12} R_1 Q_{21}^d] &= \text{tr}[R_2^\top U_{21} G_{21} U_{21}^\top R_1 Q_{21}^d] \\ &= \text{tr}[G_{21} U_{21}^\top R_1 Q_{21}^d R_2^\top U_{21}]. \end{aligned}$$

Let $U_{21}^\top R_1 Q_{21}^d R_2^\top U_{21} = \exp(z) \in \text{SO}(3)$ from Rodrigues' formula. Using the MATLAB symbolic tool, we find

$$\begin{aligned} \text{tr}[U_{12}^\top R_1 Q_{12}^d{}^\top R_2^\top U_{12} G_{12}] \\ = \cos \|z\| \sum_{i=1}^3 g_i \left(1 - \frac{z_i}{\|z\|^2}\right) + \sum_{i=1}^3 g_i \frac{x_i^2}{\|x\|^2} \leq \sum_{i=1}^3 g_i = \text{tr}[G_{21}], \end{aligned}$$

since $0 \leq \frac{z_i^2}{\|z\|^2} \leq 1$. Inserting this into (101) and recall that $\text{tr}[K_{21}] = \text{tr}[G_{21}] = k_{21}^\alpha + k_{21}^\beta$, we get

$$\|E_{\Omega_2}\| \leq \frac{1}{\sqrt{2}} \text{tr}[G_{21}] = \frac{1}{\sqrt{2}} (k_{21}^\alpha + k_{21}^\beta).$$

Substituting this into (100) leads to

$$\|\epsilon_a\| \leq \frac{1}{\sqrt{2}} (k_{21}^\alpha + k_{21}^\beta) \|e_{\Omega_2}\| + B_{\Omega_d} \|e_{21}\|. \quad (102)$$

As for ϵ_b , we can rewrite it as follows

$$\hat{e}_b = Q_{21}^d{}^\top R_1^\top \dot{K}_{21} R_2 - R_2^\top \dot{K}_{21} R_1 Q_{21}^d,$$

$$\|\hat{e}_b\| = \|(Q_{21}^d{}^\top R_1^\top \tilde{U}_{21} \tilde{G}_{21} \tilde{U}_{21}^\top R_2 - R_2^\top \tilde{U}_{21} \tilde{G}_{21} \tilde{U}_{21}^\top R_1 Q_{21}^d)\|,$$

with further rearrangement, we get

$$\begin{aligned} \|\hat{e}_b\| &= \|\tilde{U}_{21}^\top R_1 Q_{21}^d (Q_{12}^d{}^\top R_1^\top \tilde{U}_{21} \tilde{G}_{21} \tilde{U}_{21}^\top R_2 \\ &\quad - R_2^\top \tilde{U}_{21} \tilde{G}_{21} \tilde{U}_{21}^\top R_1 Q_{21}^d) Q_{12}^d{}^\top R_1^\top \tilde{U}_{21}\| \\ &= \|(\tilde{G}_{21} \tilde{U}_{21}^\top R_2 Q_{21}^d{}^\top R_1^\top \tilde{U}_{21} - \tilde{U}_{21}^\top R_1 Q_{21}^d R_2^\top \tilde{U}_{21} \tilde{G}_{21})\| \\ &\triangleq \|(\tilde{G}_{21} \tilde{U}_{21} - \tilde{U}_{21}^\top \tilde{G}_{21})\| = \|(\tilde{G}_{21} \tilde{U}_{21} - \tilde{U}_{21}^\top \tilde{G}_{21})^\vee\|, \end{aligned}$$

where we have applied $\|A\| = \|RA\| = \|AR\|$, for any matrix $A \in \mathbb{R}^{3 \times 3}$ and $R \in \text{SO}(3)$ and we also let $\tilde{U}_{12} = \tilde{U}^\top R_2 Q_{12}^d R_1^\top \tilde{U} \in \text{SO}(3)$.

Use the property (ii) of Lemma 2 that we have developed, that is, let $\epsilon_b = 2e'_P$, $\tilde{G}_{21} = F'$ and $\tilde{U}_{21} = P'$. We claim that

$$\|\epsilon_b\| \leq 2\sqrt{\frac{2\tilde{h}_2 + \tilde{h}_5}{h_5}} \|e_{21}\| \triangleq B_{21} \|e_{21}\|. \quad (103)$$

Next, inserting (102), (103) into (98) gives rise to

$$\begin{aligned}\|\dot{e}_{21}\| &\leq \|\mathbf{e}_a\| + \|\mathbf{e}_b\| \\ &= \frac{1}{\sqrt{2}}(k_{12}^\alpha + k_{12}^\beta)\|e_{\Omega_2}\| + (B_{\Omega_d} + B_{21})\|e_{21}\|,\end{aligned}\tag{104}$$

which shows (iv) directly.

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